DESIGN CRITERIA

Tubing Rigidity

Its Relation to Size Is Dramatic - But Often Misunderstood

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Many mechanical effects follow equations that contain power functions - something in the equation will vary with the square or cube, for instance, of something else. These variations are often dramatic, so people quote them a lot. Unfortunately, though, the quotes are often taken out of context.

A prime example is the effect of diameter on frame tubing. Some people will say that a tube's rigidity is proportional to the square of its diameter; some will say to the cube; and some will say to the fourth power (and, of course, some people will shun the dramatic exponents and say it's a simple direct proportion). To design any new kind of frame in a rational way, you need to know: which is it?

Some people also will quote the same relationships not for a tube's rigidity, but for its strength. Are strength and rigidity the same thing?

As it happens, most of these assertions can be correct (or approximately so), depending on the assumptions you make. Strength and rigidity are different properties though and they often vary in different ways.[1] I'll start with examples in which some of the proportions quoted above *are* correct, and then I'll go into more detail about why rigidity and strength work as they do. Finally I'll discuss a few of the implications for frame design.

Bending and Twisting

The first step of the description is to specify the kind of strength or rigidity in question. There are four common types, corresponding to the four common ways of applying a load: axial, flexural, torsional, and shear. Axial loading is lengthwise tension or compression; flexural is bending; torsional is twisting; and shear loading tends to move portions of the object crosswise past one another, similar to a stack of cards pushed sideways.

For bicycle frames the important types of rigidity are flexural and torsional.[2] Strength is rarely a problem in normal use but could be of greater concern in modified designs; again the important types would probably be flexural and torsional. (Strength does affect a frame's ability to survive an accident, of course, and the loading to be withstood in accidents seems to be mostly of a bending type.)

The following examples, then, are for flexural and torsional loading. (Conveniently, the rigidities against these two types of loading always change by equal ratios for tubing; and the strengths also change by equal ratios, but not by the same ones used for the rigidities. For example, if a change in tube design increases the flexural rigidity by 10 percent, the torsional rigidity also increases by 10 percent.) The hidden variable that lets all the different exponents be correct is, of course, the tubing wall thickness. Here are four possible permutations:

A. If both the diameter and wall thickness are multiplied by some number - call it k - then the rigidity increases by a factor of k^4 and the strength by a factor of k^3. (Meanwhile the weight for a given length increases by a factor of k^2.)

B. If the diameter is multiplied by k but the wall thickness is not changed, the rigidity increases by a factor of approximately[3] k^3 and the strength by a factor of approximately k^2. (Weight increases by a factor of approximately k.)

C. If the diameter is multiplied by k but the wall thickness is *divided* by k (so that the weight remains approximately the same) the rigidity increases by a factor of approximately k^2 and the strength by a factor of approximately k.

A fourth example is worth mentioning, even though (or because) it doesn't involve a diameter change:

D. If the diameter stays constant and the wall thickness is multiplied by k, the flexural and torsional rigidities increase by approximately the simple factor of k, and so does the strength (and the weight). For any frame design that uses standard lugs and fittings, of course, this is the only change available.

Deducing from these examples, an approximate rule of thumb would appear to be that rigidity depends on the product of the wall thickness and the cube of the diameter; and strength depends on the product of wall thickness and square of diameter.

As we'll see, this rule is a useful approximation, reasonably accurate for thin-walled tubing, but it leaves the reasons (and the exact magnitudes of change) a mystery. Also, common sense dictates that examples B and C must encounter some sort of limit.

The reasons do take some careful thought, but they aren't very complex (and they equip you to find the limits and the exact values). I'll discuss rigidity first, and then add one more consideration that will explain strength.

Moment of Inertia

The rigidity of an object can be mathematically defined as the ratio between the load applied (of some specific type) and the amount of deformation that results.[4] This ratio depends on three things:

- The stiffness (i.e., modulus of elasticity) of the material itself;
- The amount of material present to resist the deforming load; and
- The shape in which the material is arranged, which determines its mechanical advantage against the load applied.

The modulus of elasticity will depend only on the material itself. The effect of the other two factors - amount of material, and shape - is customarily expressed as a single term called the "moment of inertia" (not to be confused with bending or twisting moments; related to them only through its mathematical ancestry). In the foregoing examples, since they all assume the same kind of material, all the changes in rigidity depend on changes in this term.[5]

Strain Patterns

The moment of inertia is calculated by considering the cross section of an object as the sum of many infinitesimal portions or " elements" (Figure 1). When the object is loaded so as to deform a given amount, each element will suffer a predictable amount of deformation ("strain")[6] determined by its position within the cross section. For example, consider a tube under a bending load (Figure 2): elements on the outside of the bend are stretched, those on the inside are compressed, and the material along the " neutral axis" (located at the tube's midplane) is not deformed at all. Each element is strained by an amount proportional to its

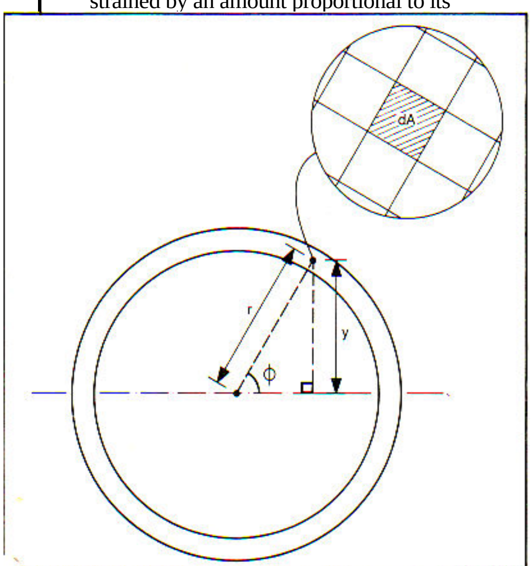

Figure 1: Element of Cross-Section Area

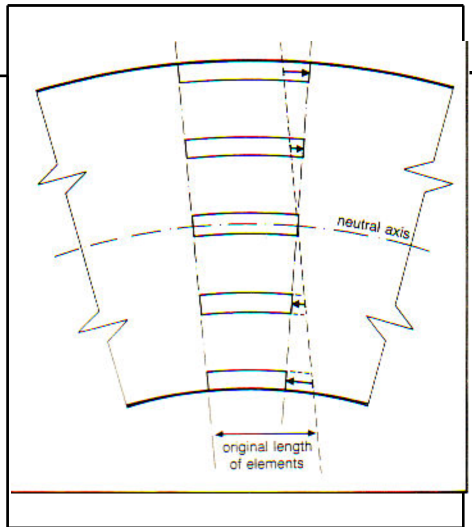

Figure 2: Relationship of Deformation to Position in Bending (exaggerated)

distance from the neutral axis.

A torsional load creates a quite different deformation pattern. Each element is deformed in shear, as adjacent segments of the tube rotate past one another (Figure 3); and instead of a plane, the neutral axis is a single line along the cylindrical axis of the tube. As with the pattern for bending, though, the strain in each element is proportional to its distance from the neutral axis. (In torsion this proportionality holds true for cylindrical tubing only.)

In either of these loading situations, each element will resist its own deformation on the microscopic level, and, on the large-scale level, contribute a related amount of resistance to the deformation of the tube as a whole. What will be the magnitude of these resistances?

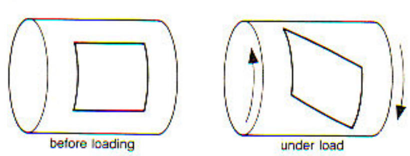

Figure 3: Shearing Deformation under Torsional Load (exaggerated)

Moment Arm - Twice

Both the small-scale and large-scale types of resistance are determined by the distance from the neutral axis (in Figure 1, distance y for bending and distance r for torsion), through the principle of moment-arm (lever) length, but somewhat differently on the two different scales:

The simpler of the two - the resistance of the element to its own deformation - is just the characteristic stress[7] that a material exerts in proportion to the amount of strain in it.[8] Since the strain is proportional to the distance from the neutral axis, so is the stress.

In the resistance to overall deformation, the distance of the element from the neutral axis comes into play an additional time. The force in each element acts to straighten (or untwist) the tube by exerting a *moment* (rather than a simple force), which tends to rotate adjacent portions of the tube back into their original positions. The *force will* be given by the element's area multiplied by its stress. The *moment* will be given by this force *multiplied by the distance from the neutral axis*.

For a given deformation and modulus of elasticity, then, the resistance offered by each element will depend on

- the area of the element; and
- the *square* of the distance from the element to the neutral axis.

The product of these terms defines the contribution of each element to the tube's moment of inertia, and the sum of them (computed as a calculus integral) is the moment of inertia itself. The value of the moment of inertia turns out to be proportional to the fourth power of the radius (or diameter) for any shape that stays the same in all its proportions while its size changes.

(Actually, it isn't necessary to perform the calculus integration to determine this proportionality. If a shape is enlarged by a factor of k and its pattern of division into elements is considered to grow with it - like words printed on an expanding balloon - then:

- Each element's linear dimensions and distance from the neutral axis will increase by a factor of k.
- Each element's area will increase by a factor of k^2.
- Each element's contribution to the moment of inertia will increase by a factor of k^4.)

In direct application, this result is only useful for tubes that retain their proportions during a change in size. But with one more step it becomes applicable to any tube at all:

One shape for which the fourth-power relationship is directly applicable is a solid cylindrical rod - if it just stays round, it retains all its proportions.

Any tube can be regarded as a large rod with a smaller one removed from its middle. Since all elements' contributions to the moment of inertia are additive, the tube's moment of inertia is the difference between that of the large rod and that of the smaller rod.

Any tube's moment of inertia, then, is proportional to the *difference in fourth powers of its outside and inside radii.*

And Strength?

A load exceeds the strength of a tube, and causes permanent deformation, when it becomes sufficient to strain some part of the tube beyond the material's elastic range, so that the stress exceeds the yield strength.

Initially you might think that strength would increase in proportion to rigidity, since rigidity is a measure of the load required to deform the object. But there's a catch:

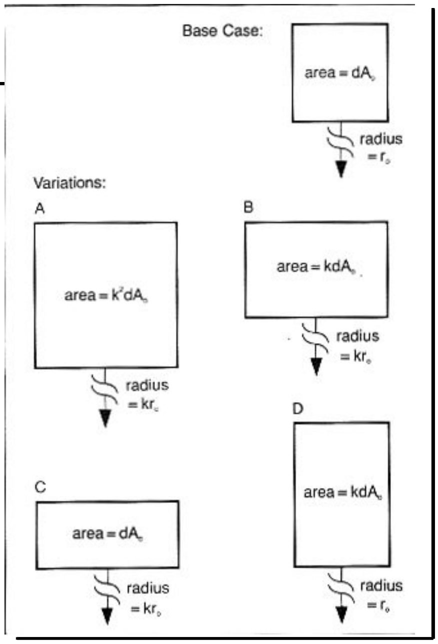

Figure 4: Change in Area of Cross-Section Elements

Rigidity does correspond to the load required to deform the shape of the tube *as a whole.* But strength depends on the maximum value of *strain produced somewhere within the cross section.* For flexure and torsion the relation between this maximum strain and the overall deformation will vary when the diameter changes, because the strain in an element is proportional to the distance of the element from the neutral axis.

As a result, the change of strength with diameter suffers a "penalty," proportional to the first power of the diameter, when compared to the change of rigidity with diameter. The mathematical term that predicts strength (as moment of inertia predicts rigidity) is called the "section modulus,"[9] and is equal to the moment of inertia divided by the distance from the neutral axis to the farthest element of the cross section. Thus, in examples A, B, and C, strength lags behind rigidity by one power of k; but in example D, where the diameter stays the same, strength keeps pace with rigidity.

Equipped with this knowledge, we can compute some exact values for the examples listed earlier. Table 1 gives values for moment of inertia, and Table 2 for section modulus, which would result if we started with a straight-gauge top tube with an outside diameter of 25.4 millimeters (1 inch) and a wall thickness of 0.8 millimeter, and enlarged it by a factor (k) of 1.5 .[10] (The identical bottom lines are not a misprint, but a predictable result of the arithmetic.)

The power-function "rules of thumb," then, turn out to be pretty good for tubes with thickness/diameter ratios in this ballpark - the worst discrepancy between the exact value (given by the difference in fourth powers of radii) and the rule of thumb value is 5.4 percent in these examples. This accuracy is actually almost as good as it is possible to try for, since variations in manufacture will produce errors approaching these: a 0.3 percent deviation from nominal diameter - commonly encountered -will produce a 1.2 percent variation in moment of inertia. (The tables carry results out to three places, but only so as to give a bit of precision in the size of the theoretical error. This three-place precision would be completely spurious to apply to actual pieces of tubing unless it were based on actual measurements of the specific pieces in question.)

If the diameter is doubled (conceivably for tandem frames or recumbents) instead of multiplied by 1.5, the error is greater, but not twice as great - for case C with the tube described, the error for k=2 is 9.3 percent. The error also worsens, though, for tubes with greater thickness/diameter ratios.

How do the rules of thumb result from the exact definitions? The approximations can be approached in either of two ways: mathematically or diagramatically.

Mathematically, as shown in the appendix, if the inner radius is expressed as the difference of outside radius and wall thickness, then the difference of fourth powers can be expressed as a series of terms. The first term of this series is the product of the wall thickness and the cube of the radius. When the thickness is small enough (compared to the radius), all the other terms become insignificant, and this first term becomes the rule of thumb.

The diagram approach depends on the division of the cross section into the array of tiny elements. If the pattern of elements is considered to keep its original arrangement while the diameter and wall thickness are changed in various ways, then the individual elements will get stretched and squashed in various ways (Figure 4), and their areas will change. The change in each element's area, when combined with the square of the change in its distance from the neutral axis, will give the "rule of thumb" result.

Implications for Bicycles

When setting out to apply all this to bicycles, one should first question whether rigidity is always desirable. It's nice for the bottom bracket to stay put and for the steering to be definite, but it's also nice, for touring at least, if the bike doesn't ride like a brick. To some extent, a frame design should strive for rigidity under lateral loads but resiliency

Table 1: Moment of Inertia

Example	0 (base case)	A	B	C	D
Diameter (mm)	25.4	38.1	38.1	38.1	25.4
Wall Thickness (mm)	0.8	1.2	0.8	0.533	1.2
Moment of Inertia I ($mm^4 \times 1000$)	4.682	23.702	16.311	11.106	6.695
Factor of Change $\frac{I}{I_o}$	—	5.063	3.484	2.372	1.430
Factor of Change predicted by simplifed power function	—	$(1.5)^4 = 5.063$	$(1.5)^3 = 3.375$	$(1.5)^2 = 2.250$	1.5
Error of simplified power function: $\frac{(simpl.) - (exact)}{(exact)}$	—	0	+ 3.2%	+ 5.4%	− 4.7%

Table 2: Section Modulus

Example	0 (base case)	A	B	C	D
Diameter (mm)	25.4	38.1	38.1	38.1	25.4
Wall Thickness (mm)	0.8	1.2	0.8	0.533	1.2
Section Modulus S $= \frac{I}{r}$ (mm^3)	368.6	1244.2	856.2	583.0	527.2
Factor of Change $\frac{S}{S_o}$	—	3.375	2.323	1.581	1.430
Factor of change predicted by simplified power function	—	$(1.5)^3 = 3.375$	$(1.5)^2 = 2.250$	1.5	1.5
Error of simplified power function: $\frac{(simpl.) - (exact)}{(exact)}$	—	0	+ 3.2%	+ 5.4%	−4.7%

under vertical ones. That's a tall order, since the geometry of the frame tends to create exactly the opposite combination. As a result, most of the vertical resiliency in a frame usually comes from the fork. Some of it, though, may come from the main "triangle," at least with certain frame shapes.

A frame's rigidity can be tailored to respond differently to different loads - such as pedaling, hard steering, and road shock - if the designer can discover the extent to which the deflection for each type of load depends on the rigidity of specific frame tubes (and on the shape of the frame). The designer can then choose an appropriate frame shape and set of tubes. This has long been an art and is gradually becoming a science. We're working on it ourselves but I won't go into it here.

The additional variable of strength can also be tailored, somewhat separately from rigidity, if a design should require it. The easiest way, of course, is to change both strength and rigidity at once by changing the wan thickness (as in example D), but in an extreme case one could actually strengthen a tube without changing the rigidity (but with a penalty in weight), by reducing the diameter and markedly thickening the wall (by a factor of roughly k^2, if the diameter were divided by k).

Down Tubes and Beer Cans

Standard present-day bicycle frames do show recognition of the effect of diameter in one aspect: the down tube and seat tube, which typically bear greater loads than the top tube - torsion in the down tube, and lateral bending in the seat tube - have a diameter $12^1/_2$ percent greater than the top tube ($1^1/_8$ inch instead of 1 inch). For a given wall thickness, this extra diameter gives the larger tubes more than 40 percent more rigidity, with only about 12½ percent more weight.

Obviously, though, there's a limit to the amount you can enlarge the diameter without thickening the wall - especially if you choose the even more tempting, constant - weight course (example C) of increasing the diameter while *thinning* the wall. Eventually the wall fails by crumpling, a failure mode called "local buckling" (also known as "the beer-can effect").

A customary rule of thumb used by engineers to avoid local buckling is that a tube's wall thickness should be no less than 1/50 of its diameter. Most high-quality steel bicycle tubing turns out to be fairly close to this limit, at least in the midsection: for 1-inch tubing the rule gives a minimum thickness of 0.51 millimeter and for $1^1/_8$-inch tubing, 0.57 millimeter. In lightweight steel tubing, then, the main frame tubes, are proportioned to have about as much strength and rigidity as possible, and to be prudent any modification to increase these qualities must include an increase in thickness and therefore in weight. A few builders consider this tradeoff worthwhile for single-rider frames. For tandems, of course, it can be emphatically worthwhile.

Room for Growth

If the tubing is not steel but aluminum (or titanium), the optimum diameter changes dramatically, because for a given weight and diameter, an aluminum tube (for instance) has roughly three times as thick a wall as a steel tube Zias. As Gary Klein has pointed out,[11] this means the aluminum tube can be enlarged and thinned (as in case C) by a factor of roughly 1.7 before it reaches the same thickness/diameter ratio as the steel tube of the same weight. This enlargement will produce multiplication of approximately (by the power-function rule of thumb) 3 in the section modulus and 5.2 in the moments of inertia. While aluminum is neither as strong nor as stiff as steel, these increases more than overcome the differences.

One part of a standard steel frame that isn't close to the 1:50 ratio, and probably could use more rigidity, is the chainstays. These members have a wall thickness comparable to that of down tubes, but their maximum diameter is rarely more than $^4/_5$ as great. Framebuilder Tom Kellogg points out that torsion and bending occur in chainstays when the bottom bracket tilts under a pedaling load. He contends that chainstays should be made in large diameters so that their additional rigidity would help hold the bottom bracket still. I agree. (Some details that might need watching, though, would be the rigidity of the rear dropouts and axle and of the chainstay-bridge attachments, so that none of these becomes fatigued by being a "weak link" attached to chainstays that are more rigid than before.)

One more issue comes to mind: aerodynamic frame tubes. For lateral rigidity, these tubes are a disaster. When a tube is squashed into a vertical oval, bringing all its material closer to the vertical plane which is the neutral axis for lateral bending, the moment of inertia about this axis takes a beating, unless the wall is thickened considera bly. (For instance, if the wall is thickened uniformly, it must be thickened by the square of the ratio between original and "squashed" diameters. Non-uniform thickening - more on the sides and less on the top and bottom - can do the job with a little less weight, but not much, and it's a considerably more specialized job.)

An oval-tubed frame, then, will either be heavier than a round-tubed one (if its walls are thicker) or whippier (if it's the same gauge). Such a frame may win a coasting contest, but a road race may be a different story. (Or it may not. I'm still looking for someone who knows what frame rigidity is worth ergonomically. Anyhow, for the sake of science, I for one would like to see a race include a bike built out of oval tubing turned *sideways.)* We haven't gotten our hands on an aero frame since we got our frame-rigidity testing machine, but when we do, we'll let you know.

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[1]Rigidity is the ratio between a loading force or moment (torque) applied and the deformation that results. Strength is the magnitude of the greatest load that the object can bear without suffering permanent deformation.

[2]Within the plane of its frame, a bicycle has a good degree of triangulation, so that it can bear most loads applied from within this plane - vertical and forward or backward loads - as axial (lengthwise) loads on the tubes, with magnitudes and resulting deflections that are fairly small. Against lateral forces, however, the frame is barely braced at all, and it must resist these forces as flexural, torsional, and shearing loads. Shearing deflections do occur, but, like axial ones, they are fairly small. The bending and twisting loadings, however, create relatively large stresses (because, unlike axial and shear loadings, they involve rotational *deflections, and therefore allow the lengths of the frame tubes to come into play as levers). Consequently they cause large deflections. Most of the deflection of a bicycle frame results from bending and twisting of the tubes in response to lateral forces.*

[3]These results would apply exactly for tubing whose inside and outside radii both changed by the same proportion. However, all these examples except A involve changes in the thickness/diameter ratio, and therefore violate this requirement. There is one loophole: the hypothetical case where the wall has no thickness, so that the inside and outside radii are equal and must therefore change at the same rate. (The only reason these examples are even approximately accurate, in fact, is that bicycle tubing happens to be fairly close to this condition.)

[4]I've attached the mathematical expressions involved at the end, for those who want to see them; but this discussion can be read without them.

[5]Actually a cylinder has two different moments of inertia - one for flexure and one for torsion - but they always change simultaneously by the same proportion, so the magnitude of change can be discussed for both at once. (Numerically, the moment of inertia for torsion - called the "polar moment of inertia" - is exactly twice the moment of inertia for bending.)

[6]Strain is the ratio of elongation (or shear displacement) to the original length (or, for shear, to the width across which the shearing takes place).

[7]Stress is the concentration of force in an element - the ratio between the force borne by the element and the cross-sectional area of that element.

[8]The ratio between stress and strain is the modulus of elasticity.

[9]The name "section modulus " is traditionally used only for the expression that applies to bending; but for cylindrical tubing the analogous expression for torsion applies. As with " moments," "section modulus" is related to " modulus of elasticity" only by their shared mathematical origin.

[10] *The lines of the tables that give actual numerical values for moment of inertia I and section modulus S depend on an equation from the appendix:*

$$I = \frac{\pi}{4}(r_o^4 - r_i^4)$$

but all the other columns could be calculated even without knowing the initial numerical value, since the coefficient $\frac{\pi}{4}$ *cancels out.*

[11] *See Gary Klein's "A Hundred Years of Monopoly: Is Steel The Ultimate Frame Material?"*, Bicycling, *September/October 1981.*

Appendix: Equations

Rigidity

Bending:

$$\textit{rigidity} = \frac{M}{K} = E\,I$$

M = bending moment applied
K = curvature = $\frac{1}{R}$
where R is radius of the curvature caused by load
E = material's modulus of elasticity
I = moment of inertia (for bending)

Torsion:

$$\textit{rigidity} = \frac{M_t}{\Theta} = G\,J$$

M_t = torsion moment applied
Θ = angle of twist (rotational displacement) per unit length
G = material's shear modulus of elasticity
J = polar moment of inertia

Moment of Inertia

Bending:

$$I = \int y^2\, dA$$
$$= \int (r \sin\phi)^2\, dA$$
$$= \int_{r_i}^{r_o}\int_0^{2\pi} r^3 (\sin\phi)^2\, d\phi\, dr$$
$$= \frac{\pi}{4}(r_o^4 - r_i^4)$$

Torsion:

$$J = \int r^2\, dA$$
$$= \int_{r_i}^{r_o}\int_0^{2\pi} r^3\, d\phi\, dr$$
$$= 2\pi\int_{r_i}^{r_o} r^3\, dr$$
$$= \frac{\pi}{2}(r_o^4 - r_i^4)$$

I = moment of inertia
J = polar moment of inertia

$\int_a^b$ = integral (calculus summation of the value of the expression following it, for all values of its variable in the range from a to b);

$$\int_0^{2\pi} (\sin\phi)^2\, d\phi = \pi$$

$$\int_{r_i}^{r_o} r^3\, dr = 1/4(r_o^4 - r_i^4)$$

$$\int_0^{2\pi} d\phi = 2\pi$$

y = distance from neutral axis (see Figure 1)
dA = area element = r dr dϕ
dr = element of radial length
rdϕ = element of circumference
ϕ = angular position of element (see Figure 1)
r_i = inside radius of tube
r_o = outside radius of tube

Section Modulus

$$S = \frac{I}{c}$$
$$= \frac{I}{r_o}$$
$$= \frac{\pi}{4}\,\frac{(r_o^4 - r_i^4)}{r_o}$$

S = section modulus
I = moment of inertia
c = maximum distance of any element from the neutral axis = r_o for tubing

Rule of Thumb Approximation

Let $T = r_o - r_i$
then $r_i = r_o - T$
and $r_i^4 = r_o^4 - 4r_o^3 T + 6r_o^2 T^2 - 4r_o T^3 + T^4$
Then $r_o^4 - r_i^4 = 4r_o^3 T - 6r_o^2 T^2 + 4r_o T^3 - T^4$
If $T << r_o$
then $6r_o^2 T^2 << 4r_o^3 T$
so $r_o^4 - r_i^4 \doteq 4r_o^3 T$

and remaining terms are even smaller; Substituting this value into the expressions for I, J, and S gives

$$I = \pi r_o^3 T$$
$$J = 2\pi r_o^3 T$$
$$S = \pi r_o^2 T$$

(It should be noted that these rules of thumb are usually more accurate to predict ratios of change than to predict actual magnitude; e.g., "$\pi r_o^3 T$" gives a figure 10 percent high for moment of inertia in case *0* of the examples.)